Pergamon

www.elsevier.com/locate/jappmathmech

J. Appl. Maths Mechs, Vol. 64, No. 6, pp. 871–890, 2000

0021-8928/00/$—see front matter

PII: S0021-8928(00)00118-0

CONTROLLABILITY AND STABILIZATION OF THE PROGRAMMED MOTIONS OF A TRANSPORT ROBOT†

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(*Received* 6 *January* 2000)

A non-linear mathematical model for the motion of a transport robot (TR) with a caterpillar chassis and with drives based on DC motors, which is a non-holonomic electromechanical system, is considered. Non-linear canonical transformations of the coordinates of the state and control space are constructed, which reduce the initial equations of motion of the TR to a simpler canonical form, which is convenient for analysing and synthesizing control systems for the TR. The conditions for the TR to be controllable as a controlled object are found. Algorithms are given for constructing programmed motions (PMs) of the TR. Stabilizing control laws are synthesized under which the PMs of the TR are asymptotically stable and transients of a specified nature are ensured. © 2001 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

Consider a TR with a caterpillar chassis, whose drives employ independently activated DC motors (DCMs), while the transmission mechanisms have absolutely rigid elements, in plane-parallel motion along a non-deformable horizontal base. Under a number of simplifying assumptions, the motion of the TR may be modelled by the following equation [1, pp. 112–116; 2]

$$\begin{aligned}
&\dot{x}_c = V_c\cos\psi_c,\quad \dot{y}_c = V_c\sin\psi_c\\
&m_0\dot{V}_c = -k_{f011}V_c - f_1 mg + (Q_{u1}+Q_{u2})/r\\
&J_z\ddot{\psi}_c = -k_{f022}\dot{\psi}_c - f_2 mg + (Q_{u2}-Q_{u1})l/r\\
&J_i\ddot{\alpha}_i + k_{f1ii}\dot{\alpha}_i + i_{pi}^{-1}\eta_{pi}^{-1}Q_{ui} = k_{mi}I_{ai}\\
&L_{ai}\dot{I}_{ai} + R_{ai}I_{ai} + k_{ei}\dot{\alpha}_i = u_{ai}\\
&\alpha_i = i_{pi}q_i;\quad i = 1,2
\end{aligned} \qquad (1.1)$$

where x_c and y_c are the coordinates of the TR's centre of mass (the mid-point of the axis of the drive wheels (sprocket wheels)) in a fixed Cartesian system of coordinates Oxy; ψ_c – the course angle – is the angle at which the TR's longitudinal axis is inclined to the Ox, axis, $V_c = r(\dot{q}_1 + \dot{q}_2)/2$ is the velocity of the TR's centre of mass in the direction of its longitudinal axis, which coincides with the tangent to the TR's trajectory of motion, i.e. V_c is the projection of the velocity vector V_c of the centre of mass (which is also directed along the TR's longitudinal axis) onto the Cx' axis of the attached (moving) system of coordinates $Cx'y'$, whose Cx' axis points from the centre of mass along the longitudinal axis of the body of the TR to the front part of the body, it is assumed that when $V_c > 0$ the TR moves in a direction which coincides with that of the Cx' axis, but if $V_c < 0$, it moves in the direction opposite to that of Cx', a dot over a symbol denotes the operation of differentiation with respect to the time t, r and $\dot{q}_1$, $\dot{q}_2$ are the radius and angular velocities of the drive wheels (sprockets) of the left and right sides of the TR, respectively, $\dot{\psi}_c = (\dot{q}_2 - \dot{q}_1)r/(2l)$ is the course angular velocity of the TR about a vertical axis passing through its centre of mass, $2l$ is the width of the track of the TR, Q_{u1} and Q_{u2} are the components of the two-dimensional vector $Q_u = \mathrm{col}(Q_{u1}, Q_{u2})$ of generalized torques Q_{u1} and Q_{u2} conveyed from the motor shafts through the transmission to the left and right driving wheels, $P_u = (Q_{u1} + Q_{u2})/r = P_{u1} + P_{u2}$ is the tractive force of the caterpillars, $P_{ui} = Q_{ui}/r$ is the tractive force of the ith caterpillar, $M_u = (Q_{u2} - Q_{u1})l/r$ is the torque generated by the tractive forces P_{u1} and P_{u2} of the caterpillars, m_0

†*Prikl. Mat. Mekh.* Vol. 64, No. 6, pp. 909–928, 2000.

$= m + 2J_0/(r^2)$ is the reduced mass of the TR, m is the mass of the TR, J_0 is the reduced moment of inertia of all rotating parts and the caterpillar on one side of the TR, J_z is the total moment of inertia of the TR about a vertical axis passing through the centre of mass, k_{f011} and f_1 are the coefficients of the drag force $F_c = -k_{f011}V_c - f_1mg$ of linear motion of the TR, g is the acceleration due to gravity, k_{f022} and f_2 are the coefficients of the drag torque $M_c = -k_{f022}\psi_c - f_2mg$ of rotational motion of the TR about a vertical axis passing through the centre of mass, α_i is the angle of rotation of the shaft of the ith motor.

$$I_a = \mathrm{col}(I_{a1}, I_{a2}) \tag{1.2}$$

is the two-dimensional vector of the currents I_{a1} and I_{a2} in the armature circuits of the DCMs, J_i is the moment of inertia of the rotor of the ith motor, k_{f1ii} is the coefficient of the drag torque of viscous friction $M_{ci} = -k_{f1ii}\alpha_i$ on the shaft of the ith motor, i_{pi} and η_{pi} are the transfer coefficient and efficiency of the ith reduction gear of the transmission, k_{mi} is the coefficient of the electromagnetic moment $M_i = k_{mi}I_{ai}$ of the ith DCM, L_{ai} and R_{ai} are the total inductance and resistance of the armature circuit of the ith DCM, k_{ei} is the coefficient of proportionality of the back emf $u_{ei} = k_{ei}\alpha_i$ of the ith DCM

$$u_a = \mathrm{col}(u_{a1}, u_{a2}) \tag{1.3}$$

is the two-dimensional vector of voltages u_{a1} and u_{a2} applied to the armature circuits of the DCM and

$$\dot{q} = \begin{Vmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Vmatrix} = \varkappa_0 \begin{Vmatrix} V_c \\ \dot{\psi}_c \end{Vmatrix}, \quad \begin{Vmatrix} P_u \\ M_u \end{Vmatrix} = \varkappa_0^* Q_u, \quad \varkappa_0 = \begin{Vmatrix} 1/r & -l/r \\ 1/r & l/r \end{Vmatrix} \tag{1.4}$$

where the asterisk denotes transposition.

Note that, since the first two equations in the system of equations of motion (1.1) of the TR describe non-holonomic constraints [3] implemented by contact between the caterpillar chassis and the supporting horizontal surface (realized by the driving [sprocket] wheels of the caterpillar chassis), it follows that model (1.1) for the dynamics of the TR is a non-holonomic electromechanical system.

Eliminating the variables Q_{u1}, Q_{u2}, α_1, α_2, from Egs (1.1) and also using notation (1.2) and (1.3) and relations (1.4), we obtain the equations of motion of the TR as a system of non-linear ordinary differential equations (ODEs) in the variables x_c, y_c, V_c, I_{a1}, I_a

$$\begin{gathered} \dot{x}_c = V_c\cos\psi_c, \; \dot{y}_c = V_c\sin\psi_c \\ \begin{Vmatrix} V_c \\ \dot{\psi}_c \end{Vmatrix}^{\bullet} = -k_f \begin{Vmatrix} V_c \\ \dot{\psi}_c \end{Vmatrix} - F_f + A^{-1}I_a \\ \dot{I}_a = L_a^{-1}\left(u_a - R_a I_a - k_e i_p \varkappa_0 \begin{Vmatrix} V_c \\ \dot{\psi}_c \end{Vmatrix}\right) \end{gathered} \tag{1.5}$$

where

$$A = \Theta_1 + \Theta_0 A_0 = \| a_{ij} \|_{i,j=1,2}, \quad A_0 = \mathrm{diag}\| m_0, J_z \|$$

$$\Theta_0 = k_m^{-1} i_p^{-1} \eta_p^{-1} [\varkappa_0^*]^{-1}, \quad \Theta_1 = k_m^{-1} J i_p \varkappa_0$$

$$k_f = A^{-1}k_m^{-1}[i_p^{-1}\eta_p^{-1}(\varkappa_0^*)^{-1}k_{f0} + k_{f1}i_p\varkappa_0] = \| k_{fij} \|_{i,j=1,2} \tag{1.6}$$

$$F_f = \mathrm{col}(F_{f1}, F_{f2}) = A^{-1}\Theta_0 \begin{Vmatrix} f_1 mg \\ f_2 mg \end{Vmatrix}$$

$A = \Theta_0$, Θ_1, kf are constant 2×2 matrices, F_j is a two-dimensional vector and A_0 and J, k_{f0}, k_{f1}, i_p, η_p, k_m, L_a, R_a, k_e, are diagonal 2×2 matrices with diagonal elements m_0, J_z and J_i, k_{f0ii}, k_{f1ii}, i_{pi}, η_{pi}, k_{mi}, L_{ai}, R_{ai}, k_{ei}, ($i = 1, 2$), respectively.

We apply non-singular linear transformations of the variables I_a (1.2) and the controls u_a using the formulae

$$\bar{I}_a = \mathrm{col}(\bar{I}_{a1}, \bar{I}_{a2}) = A^{-1} I_a \tag{1.7}$$

$$\bar{u}_a = \mathrm{col}(\bar{u}_{a1}, \bar{u}_{a2}) = A^{-1} L_a^{-1} u_a \tag{1.8}$$

We assume that the auxiliary controls $\bar{u}_{a1}$, $\bar{u}_{a2}$ are such that

$$\dot{\bar{u}}_{a1} = u_1, \quad \bar{u}_{a2} \equiv u_2 \tag{1.9}$$

where u_1 and u_2 are the components of the vector of controls

$$u = \mathrm{col}(u_1, u_2) \tag{1.10}$$

applied to the inputs of system (1.5)–(1.9).

Then the equations of motion of the TR (1.5)–(1.10) may be written as a system of non-linear ODEs

$$\dot{z} = F(z, u), \quad z_0 = z(t_0), \quad t \geqslant t_0 \tag{1.11}$$

where

$$z = \mathrm{col}(z_1, z_2, z_3, z_4) \tag{1.12}$$

$$z_1 = \mathrm{col}(x_c, y_c), \quad z_2 = \mathrm{col}(V_c, \psi_c), \quad z_3 = \mathrm{col}(\bar{I}_{a1}, \dot{\psi}_c), \quad z_4 = \mathrm{col}(\bar{u}_{a1}, \bar{I}_{a2})$$

$$F(z, u) = \mathrm{col}(F_1(z_2), \ F_2(z_2^3), \ F_3(z_2^4), \ F_4(z_2^4, u)) \tag{1.13}$$

$$z_i = \mathrm{col}(z_{i1}, z_{i2}), \quad z_i^j = \mathrm{col}(z_i, z_{i+1}, \ldots, z_j), \quad j \geqslant i; \quad z_i^i = z_i$$

$$F_1(z_2) = \mathrm{col}(z_{21} \cos z_{22}, z_{21} \sin z_{22})$$

$$F_2(z_2^3) = C_{20} + C_{22} z_{21} + D_2 z_3$$

$$F_3(z_2^4) = C_{30} + C_{32} z_{32} + C_{33} z_3 + D_3 z_4$$

$$F_4(z_2^4, u) = C_{42} z_{21} + C_{43} z_3 + C_{44} z_{42} + u$$

$$C_{20} = \left\| \begin{matrix} -F_{f1} \\ 0 \end{matrix} \right\|, \quad C_{22} = \left\| \begin{matrix} -k_{f11} \\ 0 \end{matrix} \right\|, \quad D_2 = \left\| \begin{matrix} 1 & -k_{f12} \\ 0 & 1 \end{matrix} \right\|$$

$$C_{30} = \left\| \begin{matrix} 0 \\ -F_{f2} \end{matrix} \right\|, \quad C_{32} = \left\| \begin{matrix} -\bar{k}_{e11} \\ -k_{f21} \end{matrix} \right\|, \quad C_{33} = \left\| \begin{matrix} -\bar{R}_{a11} & -\bar{k}_{e12} \\ 0 & -k_{f22} \end{matrix} \right\|$$

$$D_3 = \left\| \begin{matrix} 1 & -\bar{R}_{a12} \\ 0 & 1 \end{matrix} \right\|, \quad C_{42} = \left\| \begin{matrix} 0 \\ -\bar{k}_{e21} \end{matrix} \right\|, \quad C_{43} = \left\| \begin{matrix} 0 & 0 \\ -\bar{R}_{a21} & -\bar{k}_{e22} \end{matrix} \right\|$$

$$C_{44} = \left\| \begin{matrix} 0 \\ -\bar{R}_{a22} \end{matrix} \right\|, \quad \bar{R}_a = A^{-1} L_a^{-1} R_a A = \left\| \bar{R}_{aij} \right\|_{i,j=1,2}$$

$$\bar{k}_e = A^{-1} L_a^{-1} k_e i_p \varkappa_0 = \left\| \bar{k}_{eij} \right\|_{i,j=1,2}$$

Note that the state vector z (1.12) of system (1.11)–(1.13) is related to the state vector

$$\bar{z} = \mathrm{col}(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$$
$$\bar{z}_1 = \mathrm{col}(x_c, y_c),\ \bar{z}_2 = \mathrm{col}(V_c, \psi_c),\ \bar{z}_3 = \dot{\psi}_c,\ \bar{z}_4 = I_a = \mathrm{col}(I_{a1}, I_{a2}) \tag{1.14}$$

of the initial equations of the TR (1.5), (1.6) by a linear transformation

$$z = H_1\bar{z} + H_0 u_a \tag{1.15}$$

and the vector $\bar{z}$ is related to the vector z by a linear transformation

$$\bar{z} = H_2 z \tag{1.16}$$

where

$$H_1 = \mathrm{diag}(I_4, H_{11}),\ H_{11} = \begin{Vmatrix} 0 & a_{111} & a_{112} \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{121} & a_{122} \end{Vmatrix},\ H_0 = \begin{Vmatrix} O_{6\times 2} \\ h_1^* A^{-1} L_a^{-1} \\ O \end{Vmatrix}$$

$$H_2 = \mathrm{diag}(I_4, H_{21}),\ H_{21} = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ a_{11} & 0 & 0 & a_{12} \\ a_{21} & 0 & 0 & a_{22} \end{Vmatrix}$$

H_1, H_{11} and H_0 are 8×7, 4×3 and 8×2 matrices, respectively, I_m is the $m \times m$ identity matrix, $O_{6\times 2}$ is the 6×2 zero matrix, O is the zero vector (matrix) of suitable dimension, $h_1 = \mathrm{col}(1, 0)$ is a two-dimensional vector, $A^{-1} = A_1 = \| a_{1ij} \|_{i,j=1,2}$ and H_2 and H_{21} are 7×8 and 3×4 matrices, respectively.

We also note that, for the initial model (1.5), (1.6) of the dynamics of the TR, it follows from Eqs (1.8) and (1.9) that the components u_{a1} and u_{a2} of the vector of voltages u_a (1.3) applied to the armature circuits of the DCMs are related to the components u_1 and u_2 of the vector of controls u (1.10) by the relations

$$u_{aj} = u_{aj}(t) = L_{aj} a_{j1} \int_{t_0}^{t} u_1(\tau) d\tau + L_{aj} a_{j2} u_2(t),\ j = 1, 2,\ t \geqslant t_0 \tag{1.17}$$

and consequently the initial equations of motion (1.5), (1.6), (1.17) of the TR are equivalent, in view of formulae (1.7)–(1.9), (1.14)–(1.16), to the model of the TR dynamics represented by system (1.11)–(1.13), (1.10). In what follows, therefore, the formulation and solution of the problem will be given for the model of the TR dynamics (1.11)–(1.13), (1.10).

System (1.11)–(1.13), (1.10) is said to be controllable [4] if, for any two states $z_{p0} \in R^8$ and $z_{p1} \in R^8$ (where R^n is a Euclidean n-space) and any $t_0 < t_1$, $t_1 - t_0 < \infty$, a control $u = u(t)$ (1.10) exists such that the corresponding solution $z(t)$ (1.12) of system (1.11)–(1.13) satisfies the boundary (initial-boundary) conditions

$$z(t_0) = z_{p0},\ z(t_1) = z_{p1} \tag{1.18}$$

A solution

$$z = z_p(t),\ t \in [t_0, t_1] \tag{1.19}$$

of system (1.11)–(1.13), (1.10) satisfying boundary conditions (1.18) will be called a programmed motion (PM), and the corresponding control

$$u = u_p(t),\ t \in [t_0, t_1] \tag{1.20}$$

will be called a programmed control.

We will consider some PM $z_p(t)$ (1.19), (1.18) of system (1.11)–(1.13), (1.10). We shall say that it is stabilizable if a control law with feedback with respect to the state vector z, of the form

$$u = u(t, z),\ t \geqslant t_0 \tag{1.21}$$

exists such that the PM $z_p(t)$ (1.19), (1.18) is asymptotically stable.

The problems considered below consist of investigating the controllability and stabilizability conditions for the mathematical model (1.11)–(1.13), (1.10) of the TR as a controlled object. Algorithms will be described for constructing a PM and for synthesizing stabilizing control laws.

2. THE EQUATIONS OF MOTION OF THE TR IN CANONICAL FORM

The methods proposed below to investigate the conditions for controllability of the TR, the algorithms for constructing a PM, for synthesizing stabilizing controls and for analysing the stability of a PM are based on reducing the initial model (1.11)–(1.13), (1.10) of the TR's motion to canonical form, using non-linear transformations of the coordinates of the state and control space.

We will define a canonical form for describing the equations of motion of the TR to be their representation as a linear ODE

$$\dot{x} = Px + Qw,\ x(t_0) = x_0,\ t \geqslant t_0 \tag{2.1}$$

where

$$\begin{aligned} &x = \operatorname{col}(x_1, x_2, x_3, x_4),\ w = \operatorname{col}(w_1, w_2) \\ &x_1 = \operatorname{col}(x_c, y_c),\ \ x_i = \operatorname{col}(x_{i1}, x_{i2}) = \dot{x}_{i-1} = x_1^{(i-1)},\ \ i = 2,3,4 \end{aligned} \tag{2.2}$$

$$P = \begin{Vmatrix} O & I_6 \\ O & O \end{Vmatrix},\ \ Q = \begin{Vmatrix} O \\ I_2 \end{Vmatrix} \tag{2.3}$$

x and x_0 are the eight-dimensional vectors of the canonical state variables of the TR at the actual and initial times, respectively, $x_1^{(i)} = x_1^{(i)}(t)$ is the ith derivative with respect to t of $x_1 = x_1(t)$, $x_1^{(0)} = x_1$; $x_i = x_1^{(i)}$, w is the two-dimensional vector of "canonical" controls and P and Q are constant 8×8 and 8×2 block matrices.

3. REDUCTION OF THE EQUATIONS OF MOTION OF THE TR TO CANONICAL FORM

We will construct a transformation of coordinates in the state space z and control space u of the initial equations of motion (1.11)–(1.13), (1.10), which reduce them to the simpler canonical form (2.1)–(2.1).We will seek transformations in the form

$$x = \Psi(z) \tag{3.1}$$

$$w = \Psi_5(z_2^4, u) \tag{3.2}$$

where Ψ and Ψ_5 are eight- and two-dimensional vector functions.

$$\Psi(z) = \operatorname{col}(\Psi_1(z_1),\ \Psi_2(z_2),\ \Psi_3(z_2^3),\ \Psi_4(z_2^4)) \tag{3.3}$$

$$x_1 = \Psi_1(z_1) = z_1 \tag{3.4}$$

and Ψ_i ($i = 2, 3, 4, 5$) are two-dimensional vector-functions to be determined.

We will now describe an algorithm to determine the unknown vector functions Ψ_i ($i = 2, 3, 4, 5$). To do this, consider the identities

$$\dot{x}_1 = \dot{\Psi}_1(z_1) = \dot{z}_1,\ \ x_1^{(i)} = \dot{x}_i = \dot{\Psi}_i(z_2^i) = \sum_{k=2}^{i} \frac{\partial \Psi_i(z_2^i)}{\partial z_k} \dot{z}_k,\ \ i = 2,3,4 \tag{3.5}$$

where $\partial\Psi_i(z_2^i)/\partial z_k$ is the $m \times m$ Jacobian. Substituting into (3.5) the time derivatives $\dot{x}_i$ ($i = 1, 2, 3, 4$) along trajectories of system (2.1)–(2.3) and $\dot{z}_i$($i = 1, 2, 3, 4$) along trajectories of system (1.11)–(1.13), (1.10), we obtain the relations

$$x_2 = F_1(z_2) = \left\| \begin{matrix} z_{21}\cos z_{22} \\ z_{21}\sin z_{22} \end{matrix} \right\| \equiv \Psi_2(z_2) \tag{3.6}$$

$$x_3 = \frac{\partial\Psi_2(z_2)}{\partial z_2} F_2(z_2^3) = L_2(z_2)F_2(z_2^3) = K_3(z_2) + L_3(z_2)z_3 \equiv \Psi_3(z_2^3) \tag{3.7}$$

$$(x_2 = \dot{x}_1, x_3 = \dot{x}_2 = x_1^{(2)})$$

$$L_2(z_2) = \frac{\partial\Psi_2(z_2)}{\partial z_2} = \frac{\partial F_1(z_2)}{\partial z_2} = \left\| \begin{matrix} \cos z_{22} & -z_{21}\sin z_{22} \\ \sin z_{22} & z_{21}\cos z_{22} \end{matrix} \right\| \tag{3.8}$$

$$K_3(z_2) = L_2(z_2)(C_{20} + C_{22}z_{21}), \quad L_3(z_2) = L_2(z_2)D_2 \tag{3.9}$$

$$x_4 = \sum_{k=2}^{3} \frac{\partial\Psi_3(z_2^3)}{\partial z_k} F_k(z_2^{k+1}) = K_4(z_2^3) + L_4(z_2)z_4 \equiv \Psi_4(z_2^4) \tag{3.10}$$

$$(x_4 = \dot{x}_3 = x_1^{(3)})$$

$$K_4(z_2^3) = \frac{\partial\Psi_3(z_2^3)}{\partial z_2} F_2(z_2^3) + L_3(z_2)[C_{30} + C_{32}z_{21} + C_{33}z_3] \tag{3.11}$$

$$L_4(z_2) = \frac{\partial\Psi_3(z_2^3)}{\partial z_3} D_3 = L_3(z_2)D_3$$

$$w = \sum_{k=2}^{3} \frac{\partial\Psi_4(z_2^4)}{\partial z_k} F_k(z_2^{k+1}) + \frac{\partial\Psi_4(z_2^4)}{\partial z_4} F_4(z_2^4, u) =$$

$$= K_5(z_2^4) + L_5(z_2)u \equiv \Psi_5(z_2^4, u) \tag{3.12}$$

$$(w = \dot{x}_4 = x_1^{(4)})$$

$$K_5(z_2^4) = \sum_{k=2}^{3} \frac{\partial\Psi_4(z_2^4)}{\partial z_k} F_k(z_2^{k+1}) + L_4(z_2)(C_{42}z_{21} + C_{43}z_3 + C_{44}z_{42}) \tag{3.13}$$

$$L_5(z_2) = \frac{\partial\Psi_4(z_2^4)}{\partial z_4} = L_4(z_2)$$

We have thus constructed the initial transformations (3.1) and (3.2) in analytical form (3.1), (3.3), (3.4), (3.6)–(3.11) and (3.12), (3.13), respectively.

It will now be shown that the initial transformation just constructed, (3.1), (3.3), (3.4), (3.6)–(3.11) and (3.12), (3.13), are uniquely solvable for z and u, respectively. By virtue of (3.4), we have

$$z_1 = \Phi_1(x_1) = x_1 \tag{3.14}$$

Let us compute the principal minors Δ_1 and Δ_2 of the matrix L_2 (3.8)

$$\Delta_1 = \cos z_{22} > 0 \text{ for } z_{22} \in \Omega_{z22} = (-\pi/2, \pi/2)$$

$$\Delta_2 = z_{12} \neq 0 \quad \text{for } z_{21} \in \Omega_{z21} = \begin{cases} \Omega_{z21}^{+}, & \text{if } z_{21} = V_c > 0 \\ \Omega_{z21}^{-}, & \text{if } z_{21} = V_c < 0 \end{cases} \tag{3.15}$$

where

$$\Omega_{z21}^{+} \equiv (\varepsilon_V, k_V) \tag{3.16}$$

$$\Omega_{z21}^{-} \equiv (-k_V, -\varepsilon_V) \tag{3.17}$$

and ε_v and k_v are certain positive real numbers, $0 < \varepsilon_v < k_v < \infty$.

Throughout what follows, to fix our ideas (to avoid superfluous notation and repetition of derivations), we will consider the case in which the set Ω_{z21}, occurring in (3.15), is of the form (3.16), i.e.

$$\Omega_{z21} = \Omega_{z21}^{+} \equiv (\varepsilon_V, k_V) \tag{3.18}$$

and we introduce a certain parameter $\rho = 1$ corresponding to that case.

Note that the case in which the set Ω_{z21}, occurring in (3.15), is of the form (3.17), i.e.

$$\Omega_{z21} = \Omega_{z21}^{-} \equiv (-k_V, -\varepsilon_V) \tag{3.19}$$

is treated in exactly the same way, provided that throughout the following Sections 2–6 one replaces set (3.16) by set (3.17), set (3.18) by set (3.19), and $\rho = 1$ by $\rho = -1$. This yields estimates and propositions similar to those formulated below.

In the case when the set Ω_{z21}, occurring in (3.15), is of the form (3.18), it follows from Theorem 20.9 in [5, p. 484] that transformation (3.6) is uniquely solvable for z_2 in the rectangular domain

$$\Omega_{\Psi 2} = \{z_2 = \mathrm{col}(z_{21}, z_{22}) \in R^2 : \ z_{21} \in \Omega_{z21} \equiv \Omega_{z21}^{+}, z_{22} \in \Omega_{z22}\} \tag{3.20}$$

that is, the following inverse transformation exists

$$z_2 = \Phi_2(x_2) \tag{3.21}$$

$$\Phi_2(x_2) = \mathrm{col}(\Phi_{21}(x_2,), \Phi_{22}(x_2)) \tag{3.22}$$

$$\Phi_{21}(x_2) = \rho \cdot (x_{21}^2 + x_{22}^2)^{1/2} \equiv z_{21} = V_c \in \Omega_{z21} \equiv \Omega_{z21}^{+}, \ \rho = 1, \ x_2 \in \Omega_{\Phi_2} \tag{3.23}$$

$$\Phi_{22}(x_2) = \mathrm{arctg}(x_{22}/x_{21}) \in \Omega_{z22}, \ x_2 \in \Omega_{\Phi_2} \tag{3.24}$$

$$\begin{aligned} \Omega_{\Phi_2} = \{&x_2 = \mathrm{col}(x_{21}, x_{22}) \in R^2 : x_{21} \in \{R^1 \setminus 0\}, \ x_{22} \in R^1 \\ &z_2 = \Phi_2(x_2) \in \Omega_{\Psi_2}\} \end{aligned} \tag{3.25}$$

Furthermore, since the matrices L_2 (3.8), L_3 (3.9), L_4 (3.11) and L_5 (3.13) are such that $|\det L_i(z_2)| = |z_{21}| > \varepsilon_v > 0$ ($i = 2, 3, 4, 5$) for $z_2 \in \Omega\Psi_2$, it follows that

$$\mathrm{rank}\, L_i(z_2) = 2, \ z_2 \in \Omega_{\Psi_2}, \ i = 2,3,4,5 \tag{3.26}$$

and inverse matrices $L_i^{-1}(z_2)$ ($i = 2, 3, 4, 5$) for the corresponding values of $z_2 \in \Omega_{\Psi_2}$ exist Consequently, transformations (3.7), (3.10) and (3.12) are uniquely solvable for z_3, z_4 and u, respectively, that is, they have inverses, of the form

$$z_i = \Phi_i(x_2^i), \ i = 3,4 \tag{3.27}$$

where

$$\Phi_i(x_2^i) = M_i(x_2^{i-1}) + N_i(x_2)x_i, \ i = 3,4 \tag{3.28}$$

$$M_i(x_2^{i-1}) = -L_i^{-1}(\Phi_2(x_2))K_i(\Phi_2^{i-1}(x_2^{i-1})) \tag{3.29}$$

$$\begin{aligned} &N_i(x_2) = L_i^{-1}(\Phi_2(x_2)) = D_{i-1}^{-1}L_{i-1}^{-1}(\Phi_2(x_2)), \ i = 3,4 \\ &L_2^{-1}(\Phi_2(x_2)) = L_{2x}(x_2) = \left\| l_{2xij}(x_2) \right\|_{i,j=1,2}, \ l_{2x1j}(x_2) = \\ &= x_{2j}/\Phi_{21}(x_2), \ l_{2x2j}(x_2) = (-1)^j x_{2,3-j}/[\Phi_{21}(x_2)]^2, \ j = 1,2 \\ &\Phi_2^{i-1}(x_2^{i-1}) = \mathrm{col}(\Phi_2(x_2), \ \Phi_3(x_2^3), \ldots, \ \Phi_{i-1}(x_2^{i-1})) \end{aligned} \tag{3.30}$$

$$u = \Phi_5(x_2^4, w) \tag{3.31}$$

$$\Phi_5(x_2^4, w) = M_5(x_2^4) + N_5(x_2)w \tag{3.32}$$

$$M_5(x_2^4) = -L_5^{-1}(\Phi_2(x_2))K_5(\Phi_2^4(x_2^4)), \quad N_5(x_2) = L_5^{-1}(\Phi_2(x_2)) \tag{3.33}$$

Thus, taking Eqs (3.14), (3.21)–(3.30) into account, we have constructed the one-to-one inverse of the initial transformation (3.1), (3.4), (3.6)–(3.11)

$$z = \Phi(x), x \in \Omega_\Phi \tag{3.34}$$

where

$$\Phi(x) = \mathrm{col}(\Phi_1(x_1),\ \Phi_2(x_2),\ \Phi_3(x_2^3),\ \Phi_4(x_2^4)) \tag{3.35}$$

Φ_i ($i = 1, \dots, 4$) are the vector functions (3.14), (3.21)–(3.30), and

$$\Omega_\Phi = \{x \in R^8 : z = \Phi(x) \in \Omega_\Psi\} \tag{3.36}$$

$$\Omega_\Psi = \{z = \mathrm{col}(z_1, z_2, z_3, z_4) \in R^8 : z_i \in R^2, i = 1,3,4; z_2 \in \Omega_{\Psi_2}\} \tag{3.37}$$

We will now show that, if one takes any solution $x_1(t)$ of the ODE

$$x_1^{(4)} = \Psi_5(\Phi_2^4(\dot{x}_1, x_1^{(2)}, x_1^{(3)}), u) \tag{3.38}$$

which is equivalent to system (2.1)–(2.3) for $w = \Psi_5(\Phi_2^4(x_1, x_1^{(2)}, x_1^{(3)}), u)$, where $x_2^4 = \mathrm{col}(x_2, x_3, x_4) = \mathrm{col}(x_1, x_1^{(2)}, x_1^{(3)})$, substitutes it into system (3.5)

$$x_1^{(i)} = \dot{x}_i = \dot{\Psi}_i(z_2^i) = \Psi_{i+1}(z_2^{i+1}) = x_{i+1}, \quad i = 1,2,3 \tag{3.39}$$

where $z_2^1 = z_2$, and uses this system to define the vector functions $z_i(t)$ ($i = 2, 3, 4$), then the system of vector functions

$$x_1(t) = z_1(t), z_2(t), z_3(t), z_4(t) \tag{3.40}$$

will be a solution of system (1.11)–(1.13), (1.10).

Substituting the system of vector functions (3.40) into system (1.11)–(1.13), (1.10), all the equations of the latter become identities; in particular, we obtain the identity

$$\dot{x}_1 = \dot{z}_1 \equiv F_1(z_2) \tag{3.41}$$

Differentiating this identity with respect to t, we obtain

$$\ddot{x}_1 = \ddot{z}_1 = \dot{x}_2 = \dot{\Psi}_2(z_2) = \frac{\partial \Psi_2(z_2)}{\partial z_2}\dot{z}_2 \tag{3.42}$$

It is not yet possible to replace $\dot{z}_2$ by the vector function F_2, since we have not yet shown that the vector functions $x_1(t), z_2(t), z_3(t), z_4(t)$ (3.40) derived in this way from Eq. (3.38) and system (3.39) indeed satisfy the initial system of equations (1.11)–(1.13), (1.10); indeed, that is just what has to be proved.

Subtracting identity (3.7) term by term from (3.42), we obtain

$$\frac{\partial \Psi_2(z_2)}{\partial z_2}(\dot{z}_2 - F_2(z_2^3)) \equiv 0 \tag{3.43}$$

Similarly, differentiating the identities $x_i = \Psi_i(z_2^i)$ ($i = 3,4$) (3.39) with respect to t

$$\dot{x}_i = \dot{\Psi}_i(z_2^i) = \sum_{k=2}^{i} \frac{\partial \Psi_i(z_2^i)}{\partial z_k}\dot{z}_k, \quad i = 3,4$$

and subtracting the identities

$$\dot{x}_i = \dot{\Psi}_i(z_2^i) = \sum_{k=2}^{i} \frac{\partial \Psi_i(z_2^i)}{\partial z_k} F_k(z_2^{k+1}), \; i = 3,4$$

from (3.10) and (3.12), respectively, we obtain

$$\sum_{k=2}^{i} \frac{\partial \Psi_i(z_2^i)}{\partial z_k}(\dot{z}_k - F_k(z_2^{k+1})) \equiv 0, \; i = 3,4 \tag{3.44}$$

Equations (3.43) and (3.44) may be written as a system of equations in the unknowns $(z_k - F_k(z_2^{k+1}))$ $(k = 2, 3, 4)$

$$J_0(z_2^3)(\dot{z}_2^4 - F_2^4(z_2^5)) = 0 \tag{3.45}$$

where $z_2^5 = \mathrm{col}(z_2, \ldots, z_5)$, $z_5 = u$, $F_2^4(z_2^5) = \mathrm{col}(F_2(z_2^3), F_3(z_2^4), F_4(z_2^5))$

$$J_0(z_2^3) = \frac{\partial \Psi_2^4(z_2^4)}{\partial z_2^4} \tag{3.46}$$

is the 6×6 Jacobian and $\Psi_2^4(z_2^4) = \mathrm{col}(\Psi(z_2), \Psi_3(z_2^3), \Psi_4(z_2^4))$.

Taking relations (3.6)–(3.11) into consideration, we conclude that the matrix function J_0 (3.46) is a lower block-diagonal matrix with diagonal 2×2 blocks L_i $(i = 2, 3, 4)$ (3.8), (3.9), (3.11), which, by (3.26), are non-singular. Therefore

$$\mathrm{rank}\, J_0(z_2^3) = 6, \;\; \forall z_2^3 \in \Omega_{J0} \tag{3.47}$$

where

$$\Omega_{J0} = \{z_2^3 = \mathrm{col}(z_2, z_3) \in R^4 : z_2 \in \Omega_{\Psi_2}, z_3 \in R^2\} \tag{3.48}$$

and, consequently, taking note of (3.47), we conclude that the matrix function J_0 (3.46) is also non-singular. Hence, it follows that at each point of the set Ω_{J0} (3.48), system (3.45) has only the trivial solution

$$\dot{z}_2^4 - F_2^4(z_2^5) = 0$$

Bearing identity (3.41) in mind, we conclude that the vector function $y = \mathrm{col}(x_1, z_2, z_3, z_4) \equiv z$ is a solution of the initial system of equations (1.11)–(1.13), (1.10).

4. CONTROLLABILITY AND AN ALGORITHM FOR CONSTRUCTING A PM OF THE TR

We will first show that the model of the TR's motion in canonical form (2.1)–(2.3) is completely controllable [6, p. 269]. Since the matrix

$$S = \|Q, PQ, \ldots, P^7Q\| \tag{4.1}$$

has a submatrix $S_0 = \| Q, PQ, P^2Q, P^3Q\|$, such that, by (2.3), $|\det S_0| = 1$ and consequently

$$\mathrm{rank}\, S = \mathrm{rank}\, S_0 = 8 \tag{4.2}$$

it follows that system (2.1)–(2.3) is completely controllable [6, p. 269, Theorem 3.1], i.e., a control law

$$w = w_p = w_p(t) = Q^* e^{P^*(t_1 - t)} K_0^{-1}(x_{p1} - e^{PT} x_{p0}) \tag{4.3}$$

exists where

$$K_0 = \int_{t_0}^{t_1} e^{P(t_1-t)} QQ^* e^{P^*(t_1-t)} dt \tag{4.4}$$

is a constant positive definite 8 × 8 matrix by virtue of the complete controllability of system (2.1)–(2.3) [6], which steers system (2.1)–(2.3) from any initial state $x_p(t_0) = x_{p0} = \Psi(z_{p0}) \in R^8$ (in particular, for $z_{p0} \in \Omega_\Psi$, where Ω_Ψ is set (3.37)) to an arbitrary terminal state $x_p(t_1) = x_{p1} = \Psi(z_{p1}) \in R^8$ (in particular, for $z_{p1} \in \Omega_\Psi$) in time $t_1 - t_0 < \infty$ along a trajectory

$$x_p = x_p(t) = e^{P(t-t_0)} x_{p0} + \int_{t_0}^{t} e^{P(t-s)} Q w_p(s) ds, \quad t \in [t_0, t_1] \tag{4.5}$$

Note that in order to compute $e^{P(t-t_0)}$, $e^{P(t-s)}$, $e^{P(t_1-t)}$, e^{PT} where P is matrix (2.3), one can use the representation of $e^{P\tau}$ as

$$e^{P\tau} = \sum_{i=0}^{3} \frac{P^i \tau^i}{i!}$$

Hence, using transformations (3.6) and (3.8)–(3.10), we conclude that the control law

$$u = u_p = \Phi_5(x_{p2}^4, w_p) = \Phi_5(\Psi_2^4(z_{p2}^4), w_p) \tag{4.6}$$

where $\Psi_2^4(z_{p2}^4) = \text{col}\,(\Psi_2(z_{p2}), \Psi_3(z_{p2}^3), \Psi_4(z_{p2}^4))$, and w_p and x_p are defined by (4.3)–(4.5), steers the initial model of the motion of the TR (1.11)–(1.13) from any initial state $z_p(t_0) = z_{p0} \in \Omega_\Psi$ to an arbitrary terminal state $z_p(t_1) = z_{p1} \in \Omega_\Psi$, where Ω_Ψ is set (3.37), in time $t_1 - t_0 < \infty$ along a trajectory

$$z = z_p = \Phi(x_p)\ t \in [t_0, t_1] \tag{4.7}$$

Therefore, the initial model (1.11)–(1.13), (1.10) for the TR's motion is also controllable.

5. STABILIZABILITY CRITERIA FOR A PM OF THE TR

We will first consider the problem of synthesizing stabilizing control laws w and analysing the stability of a PM $x_p(t)$ in the set Ω_Φ (3.36), $t \geqslant t_0$, for the canonical model (2.1)–(2.3) of the TR's motion.

The fact that this model is completely controllable (i.e., that relations (4.2) and (4.3) are satisfied) implies [6, p. 274, Theorem 4.1] that a constant 2 × 8 matrix of amplification factors

$$\Gamma_0 = \|\Gamma_{01}, \ldots, \Gamma_{04}\| \tag{5.1}$$

exists where Γ_{0j} ($j = 1, \ldots, 4$) are 2 × 2 blocks, such that the matrix

$$\Gamma = P + Q\Gamma_0 \tag{5.2}$$

has given eigenvalues λ_i ($i = 1, \ldots, 8$), in particular, for example, such that the matrix Γ is stable (Hurwitzian) [16, p. 597], that is, Re $\lambda_i < 0$ ($i = 1, \ldots, 8$). In addition, the matrix Γ_0 (5.1) may be chosen so that the matrix Γ (5.2) has, say, given distinct, real, negative eigenvalues, that is

$$\lambda_i < 0 \quad (\lambda_i \neq \lambda_j,\ i \neq j;\ i = 1, \ldots, 8;\ j = 1, \ldots, 8) \tag{5.3}$$

We will synthesize a control law with "canonical" feedback with respect to x, in the form

$$w = w_p + \Gamma_0(x - x_p) \tag{5.4}$$

Then the equation of the transients $e_x = x - x_p$ in the closed system (2.1)–(2.3), (5.4), (5.3) will have the form

$$\dot{e}_x = \Gamma e_x, \quad e_x(t_0) = e_{x0}, \quad t \geqslant t_0 \tag{5.5}$$

Consequently, a PM $x_p(t)$ (4.5) of system (2.1)–(2.3), (5.4), (5.1)–(5.3) is asymptotically stable in the large, with an estimate

$$|e_x(t)| \leqslant \beta_0 |e_x(t_0)| \exp[\gamma_0(t - t_0)], \quad e_x(t_0) = e_{x0}, \quad t \geqslant t_0 \tag{5.6}$$

and the damping of the transient $e_x(t)$ will be of a given aperiodic nature (in particular, for e_{x0} such that $e_{x0} + x_{p0} = x_0 = x(t_0) \in \Omega_\Phi$), where

$$\gamma_0 = \max_i \lambda_i, \quad \lambda_i < 0 \quad (i = 1, \ldots, 8); \quad \beta_0 = \sum_{i=1}^{8} |\overline{\Gamma}_i| > 0$$

$$\overline{\Gamma}_i = \left[\prod_{\substack{k=1 \\ k \neq i}}^{8} (\Gamma - \lambda_k I_8)\right] \left[\prod_{\substack{k=1 \\ k \neq i}}^{8} (\lambda_i - \lambda_k)\right]^{-1} \quad (i = 1, \ldots, 8)$$

are the coefficient matrices of the Lagrange–Sylvester interpolation polynomial [7, p. 49]

$$e^{\Gamma(t - t_0)} = \sum_{i=1}^{8} \overline{\Gamma}_i \exp[\lambda_i(t - t_0)]$$

and

$$|a| = (a_1^2 + \ldots + a_n^2)^{1/2} \quad \text{and} \quad |A| = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$$

are the Euclidean norms (moduli) of the vector $a = \mathrm{col}(a_1, \ldots, a_n) \in R^n$ and $n \times n$ matrix

$$A = \|a_{ij}\|_{i,j=1,\ldots,n}.$$

Substituting relations (5.4), (5.1)–(5.3) into (3.31) and using the coordinate transformations of the state space (3.3)–(3.4), (3.6)–(3.11), we obtain the desired stabilizing control law with feedback with respect to z

$$\begin{gathered} u = \mathrm{col}(u_1, u_2) = \Phi_5(x_2^4, w_p + \Gamma_0(x - x_p)) = \\ = \Phi_5(\Psi_2^4(z_2^4), w_p + \Gamma_0(\Psi(z) - \Psi(z_p))) = \mathrm{col}(\overline{\Phi}_{51}(\Gamma_0, t, z), \overline{\Phi}_{52}(\Gamma_0, t, z)) \\ \left(u_i = \overline{\Phi}_{5i}(\Gamma_0, t, z) = h_i^* \Phi_5(\Psi_2^4(z_2^4), w_p + \Gamma_0(\Psi(z) - \Psi(z_p))), \quad i = 1, 2\right) \end{gathered} \tag{5.7}$$

where $h_1 = \mathrm{col}(1, 0)$, $h_2 = \mathrm{col}(0, 1)$ are two-dimensional vectors, for the initial model of the TR's motion (1.11)–(1.13).

The equation of the transient $e = z - z_p$ in the initial closed model of the motion of the TR (1.11)–(1.13), (5.7), (5.1)–(5.3) has the form

$$\dot{e} = F_e(e, t), \quad e(t_0) = e_0, \quad t \geqslant t_0 \tag{5.8}$$

where

$$F_e(e, t) = F\left(e + z_p, \Phi_5(\Psi_2^4(e_2^4 + z_{p2}^4), w_p + \Gamma_0(\Psi(e + z_p) - \Psi(z_p)))\right) - F(z_p, u_p) \tag{5.9}$$

$$F_e(0, t) \equiv 0, \quad e_0 + z_{p0} = z_0 \in \Omega_\Psi$$

Let us estimate the transient e in (5.8) and (5.9). We will assume that

$$z_p(t) \in \Omega_{zp}, \quad t \geqslant t_0 \tag{5.10}$$

where

$$\Omega_{zp} = \{z_p = \mathrm{col}(z_{p1}, z_{p2}, z_{p3}, z_{p4}) \in \Omega_\Psi : \; k_{zpik} = $$
$$= \sup_{t \geq t_0} |z_{pik}(t)| < \infty \quad (i = 3, 4; \; k = 1, 2)\} \tag{5.11}$$

Ω_Ψ is set (3.37) and k_{zpik} (i = 3, 4; k = 1, 2) are certain constants.

Using the formulae for finite increments of vector functions [8, p. 122, Lemma 3.1], relations (5.10), (5.11), (3.36), (3.37), (3.20) and (3.25), the inequality $a^2 + b^2 \geq 2^{ab}$ and the estimates

$$|a| \leq \sum_{i=1}^{n} |a_i|, \quad |A| \leq \sum_{i=1}^{n} \sum_{j=1}^{m} |a_{ij}|$$

for the moduli of the vector $a \in R^n$ and the $n \times m$ matrix A, one can show successively (see the Appendix) that the moduli $|e_i|$ (i = 1, 2, 3, 4) of the subvectors $e_i(t)$ (i = 1, 2, 3, 4) of the vector $e = z(t) - zp(t) = \mathrm{col}(e_1, e_2, e_3, e_4)$ satisfy estimates (6.1), (6.2), (6.17) and (6.25), respectively, that is

$$|e_1(t)| = |z_1(t) - z_{p1}(t)| = |e_{x1}(t)| = |x_1(t) - x_{p1}(t)| \tag{5.12}$$

$$|e_2(t)| = |z_2(t) - z_{p2}(t)| \leq \nu_2 |e_{x2}(t)|, \quad t \geq t_0 \tag{5.13}$$

where

$$\nu_2 = (\rho^2 + \varepsilon_V^{-2})^{1/2} > 0, \; \rho = 1,$$

$$|e_3(t)| \leq \nu_3(|e_{x2}(t)| + |e_{x3}(t)|), \quad t \geq t_0 \tag{5.14}$$

$$|e_4(t)| \leq \nu_4(|e_{x2}| + |e_{x3}| + |e_{x4}| + |e_{x2}|^2 + |e_{x3}|^2), \quad t \geq t_0 \tag{5.15}$$

Here ν_3 and ν_4 are certain constants, which will be defined in (6.18) and (6.26).

We now estimate $|e(t)| = |z(t) - z_p(t)|$, using estimates (5.12)–(5.15) for the vectors $e_i(t) = z_i(t) - z_{pi}(t)$ (i = 2, 3, 4). We obtain

$$|e(t)| = |\Delta_\Phi(e_x, t)| = |\Phi(e_x + x_p) - \Phi(x_p)| \leq$$
$$\leq \sum_{i=1}^{4} |e_i(t)| \leq |e_{x1}(t)| + \nu_2 |e_{x2}(t)| + \nu_3(|e_{x2}(t)| + |e_{x3}(t)|) +$$
$$+ \nu_4(|e_{x2}| + |e_{x3}| + |e_{x4}| + |e_{x2}|^2 + |e_{x3}|^2) \leq \Delta\Phi_0(e_x) =$$
$$= \left[\int_0^1 J_{\Delta\Phi 0}(\theta e_x) d\theta\right] e_x \leq \left|\left[\int_0^1 |J_{\Delta\Phi 0}(\theta e_x)| d\theta\right]\right| |e_x| \leq \mu_0 |e_x(t)| =$$
$$= \mu_0 \beta_0 |e_x(t_0)| \exp[\gamma_0(t - t_0)] = \mu |\Delta\Psi(e_0, t_0)| \exp[\gamma_0(t - t_0)]$$
$$e_0 + z_{p0} = z_0 \in \Omega_\Psi, \quad t \geq t_0 \tag{5.16}$$

where

$$\Delta\Phi_0(e_x) = \nu_{01}|e_x| + \nu_4(|e_{x2}|^2 + |e_{x3}|^2)$$
$$\nu_{01} = 4\max\{1, \nu_2 + \nu_3 + \nu_4, \nu_3 + \nu_4, \nu_4\} \tag{5.17}$$

$$J_{\Delta\Phi 0}(e_x) = \frac{\partial \Delta\Phi_0(e_x)}{\partial e_x} = \Big\| \nu_{01}|e_x|^{-1/2} e_{x1},\ \left(\nu_{01}|e_x|^{-1/2} + 2\nu_4\right)e_{x2}$$

$$\left(\nu_{01}|e_x|^{-1/2} + 2\nu_4\right)e_{x3},\ \nu_{01}|e_x|^{-1/2} e_{x4}\Big\| \tag{5.18}$$

$$\sup_{\tilde{e}_x \in [0, e_x],\ t \geq t_0} |J_{\Delta\Phi 0}(\tilde{e}_x)| \leq \nu_{01} + 2\nu_4 \sum_{i=2}^{3} |\tilde{e}_{xi}| = \mu_0,\quad \mu = \mu_0\beta_0 \tag{5.19}$$

$$[0, e_x] = \{\tilde{e}_x | \tilde{e}_x = \theta e_x,\ e_x + x_p = x \in \Omega_\Phi,\ 0 \leq \theta \leq 1\}$$

$\tilde{e}_x = \mathrm{col}(\tilde{e}_{x1}, \ldots, \tilde{e}_{x4})$; $\tilde{e}_{xi}$ $(i = 1, \ldots, 4)$ are two-dimensional vectors and $\Delta\Psi(e_0, t_0) = \Psi(e_0 + z_{p0}) - \Psi(z_{p0})$.

If follows from (5.16) and (5.17) that

$$|e(t)| = |\Delta\Phi(e_x, t)| \leq \Delta\Phi_0(e_x) = \nu_{01}|e_x| + \nu_4\left(|e_{x2}|^2 + |e_{x3}|^2\right), \tag{5.20}$$

from which it follows that the vector function $\Delta\Phi(e_x, t)$ is continuous with respect to e_x at $e_x = 0$, uniformly in $t \in [t_0, \infty]$, and moreover $\Delta\Phi(0, t) = 0$.

It follows from (5.16)–(5.19), (5.6), (5.3) that

$$|e(t)| \to 0 \quad \text{as} \quad t \to +\infty \tag{5.21}$$

We will show that Lyapunov stability of the solution $e_x = 0$ of system (5.5), (5.1)–(5.3) implies Lyapunov stability of the solution $e = 0$ of system (5.9), (5.1)–(5.3).

Take any $\varepsilon > 0$. Since the vector function $\Delta\Phi(e_x, t) = \Phi(e_x + x_{p0}(t) - \Phi(x_p(t))$ is continuous with respect to e_x at $e_x = 0$, uniformly in $t \in [t_0, \infty]$, it follows that, given $\varepsilon > 0$, one can find $\varepsilon_0 > 0$ such that

$$|e_x| < \varepsilon_0 \Rightarrow |e| = |\Delta\Phi(e_x, t)| < \varepsilon \tag{5.22}$$

where $e_x = e_x(e_{x0}, t)$ $e = e_x(e_0, t)$.

Since the solution $e_x = 0$ of system (5.5), (5.1)–(5.3) is Lyapunov stable, it follows that, given $\varepsilon_0 > 0$, one can find $\delta_0 > 0$ such that

$$|e_{x0}| < \delta_0 \Rightarrow |e_x(e_{x0}, t)| < \varepsilon_0,\quad t \in [t_0, \infty) \tag{5.23}$$

Consider the vector function

$$\Delta\Psi(e, t) = \Psi(e + z_p(t)) - \Psi(z_p(t))$$

where Ψ is the vector function (3.3), (3.4), (3.6)–(3.11). Using the continuity with respect to e at $e = 0$ of the vector function $\Delta\Psi(e, t_0)$, given $\delta_0 > 0$, one can find $\delta > 0$ such that

$$|e_0| < \delta \Rightarrow |e_{x0}| = |\Delta\Psi(e_0, t_0)| < \delta_0 \tag{5.24}$$

Taking inequalities (5.22)–(5.24) into consideration, we obtain

$$|e_0| < \delta \Rightarrow |e_{x0}| < \delta_0 \Rightarrow |e_x(e_{x,0}, t)| < \varepsilon_0 \Rightarrow |e(e_0, t)| < \varepsilon,\quad t \geq t_0$$

Consequently, it follows from the Lyapunov stability of the trivial solution of system (5.5), (5.1)–(5.3) that the trivial solution of system (5.8), (5.9), (5.1)–(5.3) is also Lyapunov stable.

We have thus shown the following.

Theorem. Let $z_p(t)$ be a given (or constructed) PM (5.10), 5.11) of the initial model of TR motion (1.11)–(1.13), (1.10). Then the stabilizing control Law u (5.7), (5.1)–(5.3) with feedback with respect to z guarantees asymptotic stability of the PM $z_p(t)$ (5.10), (5.11), and the transient $e(t) = z(t) - z_p(t)$in the closed initial model of TR motion (1.11)–(1.13), (5.7), (5.1)–(5.3) satisfies estimate (5.16)–(5.19). Note that, substituting the control law u (5.7), (5.1)–(5.3) into (1.17) and using estimates (5.16)–(5.19) for $e = z - z_p$, that follow from the theorem proved above, we obtain stabilizing laws for the variation of the control voltages

$$u_{aj} = u_{aj}(t) = L_{aj}a_{j1}\int_{t_0}^{t}\overline{\Phi}_{51}(\Gamma_0,\tau,z)d\tau + \\ + L_{aj}a_{j2}\overline{\Phi}_{52}(\Gamma_0,t,z), \quad t \geq t_0, \quad j = 1,2 \tag{5.25}$$

applied to the armature circuits of the DCMs, such that the PM

$$\bar{z}_p = H_2 z_p$$

where H_2 is the matrix defined by transformation (1.16) of the initial equations of motion of the TR (1.5), (1.6), is asymptotically stable; the transient

$$\bar{e} = \bar{z} - \bar{z}_p = H_2(z - z_p) = H_2 e$$

satisfies an estimate

$$|\bar{e}| \leq |H_2||e| \leq \bar{\mu}|\Delta\Psi(e_0,t_0)|\exp[\gamma_0(t-t_0)], \quad e_0 + z_{p0} = z_0 \in \Omega_\Psi, \quad t \geq t_0$$

where

$$\bar{\mu} = |H_2|\mu, \quad e_0 = H_1\bar{e}_0 + H_0(u_{a0} - u_{ap0}) = H_1\bar{e}_0 + H_0(u_{a0} - L_a A\bar{u}_{ap0})$$

$u_{a0} = u_a(0)$, $u_{ap0} = u_{ap}(0)$, $\bar{u}_{ap0} = \bar{u}_{ap}(0)$. The matrices H_1 and H_0 are defined by transformation (1.15).

6. APPENDIX

We will successively estimate the moduli $|e_i|$ (i = 1, 2, 3, 4) of the subvectors $e_i(t)$ of the vector $e = z(t) - z_p(t) = \mathrm{col}\,(e_1, e_2, e_3, e_4)$. We have

$$|e_1(t)| = |z_1(t) - z_{p1}(t)| = |e_{x1}(t)| = |x_1(t) - x_{p1}(t)| \tag{6.1}$$

Using the formula for finite increments of a vector-valued function

$$\Delta\Phi_2(e_{x2},t) = \Phi_2(e_{x2} + x_{p2}) - \Phi_2(x_{p2})$$

We obtain [8, p. 122, Lemma 3.1]

$$|e_2(t)| = |z_2(t) - z_{p2}(t)| = |\Delta\Phi_2(e_{x2},t)| = \\ = |\Phi_2(e_{x2} + x_{p2}) - \Phi_2(x_{p2})| = |\mathrm{col}(\Delta\Phi_{21}(e_{x2},t),\ \Delta\Phi_{22}(e_{x2},t))| = \\ = \left|\left[\int_0^1 J_{\Delta\Phi 2}(\theta e_{x2}(t),t)d\theta\right]e_{x2}(t)\right| \leq \nu_2|e_{x2}(t)|, \quad t \geq t_0 \tag{6.2}$$

Here, taking note of relations (3.23)–(3.26) and (3.30), we have

$$\Delta\Phi_{21}(e_{x2},t)=\Phi_{21}(e_{x2}+x_{p2})-\Phi_{21}(x_{p2})=$$
$$=\rho\left\{\left[(e_{x21}+x_{p21})^2+(e_{x22}+x_{p22})^2\right]^{1/2}-\left[x_{p21}^2+x_{p22}^2\right]^{1/2}\right\},$$
$$\rho=1,\quad e_{x2}+x_{p2}=x_2\in\Omega_{\Phi_2} \tag{6.3}$$
$$\Delta\Phi_{22}(e_{x2},t)=\Phi_{22}(e_{x2}+x_{p2})-\Phi_{22}(x_{p2})=$$
$$=\operatorname{arctg}\frac{e_{x22}+x_{p22}}{e_{x21}+x_{p21}}-\operatorname{arctg}\frac{x_{p22}}{x_{p21}},\quad e_{x2}+x_{p2}=x_2\in\Omega_{\Phi_2} \tag{6.4}$$

$$J_{\Delta\Phi2}(e_{x2},t)=\frac{\partial\Delta\Phi_2(e_{x2},t)}{\partial e_{x2}}=\begin{Vmatrix}\rho l_{2x11}(x_2) & \rho l_{2x12}(x_2)\\ l_{2x21}(x_2) & l_{2x22}(x_2)\end{Vmatrix}=$$
$$=\begin{Vmatrix}\rho\cos z_{22} & \rho\sin z_{22}\\ -\dfrac{\sin z_{22}}{z_{21}} & \dfrac{\cos z_{22}}{z_{21}}\end{Vmatrix},\quad \rho=1 \tag{6.5}$$

where

$$\sup_{\bar e_{x2}\in[0,e_{x2}],\ t\geqslant t_0}\left|J_{\Delta\Phi2}(\bar e_{x2},t)\right|\leqslant \nu_2 \tag{6.6}$$
$$[0,e_{x2}]=\left\{\bar e_{x2}\,\middle|\,\bar e_{x2}=\theta e_{x2},\ e_{x2}+x_{p2}=x_2\in\Omega_{\Phi_2},\ 0\leqslant\theta\leqslant 1\right\}$$
$$\nu_2=(\rho^2+\varepsilon_V^{-2})^{1/2}>0,\quad \rho=1.$$

Let us estimate $|e_i(t)|$ $(i = 3, 4)$. We have

$$|e_i(t)|=|z_i(t)-z_{pi}(t)|=\left|\Delta\Phi_i(e_{x2}^i,t)\right|=\left|\Phi_i(e_{x2}^i+x_{p2}^i)-\Phi_i(x_{p2}^i)\right|\leqslant$$
$$\leqslant\left|\Delta M_i(e_{x2}^{i-1},t)\right|+\left|\Delta N_i(e_{x2},t)x_{pi}\right|+\left|N_i(e_{x2}+x_{p2})e_{xi}\right|,\quad i=3,4 \tag{6.7}$$

Here (taking note of relations (3.28), (3.29), (3.9)–(3.11) and (6.5))

$$\Delta M_3(e_{x2},t)=M_3(e_{x2}+x_{p2})-M_3(x_{p2})=-D_2^{-1}C_{22}\Delta\Phi_{21}(e_{x2},t)=-D_2^{-1}C_{22}e_{21}$$
$$\Delta M_4(e_{x2}^3,t)=M_4(e_{x2}^3+x_{p2}^3)-M_4(x_{p2}^3)=-L_4^{-1}(\Phi_2(e_{x2}+x_{p2}))\times \tag{6.8}$$
$$\times K_4(\Phi_2^3(e_{x2}^3+x_{p2}^3))+L_4^{-1}(\Phi_2(x_{p2}))K_4(\Phi_2^3(x_{p2}^3))=$$
$$=-L_4^{-1}(e_2+z_{p2})K_4(e_2^3+z_{p2}^3)+L_4^{-1}(z_{p2})K_4(z_{p2}^3)=\mu_1(e_2^3,t)+\mu_2(e_2^3,t)+\mu_3(e_{21},e_3)$$
$$\mu_1(e_2^3,t)=-D_3^{-1}D_2^{-1}G_0(e_2^3+z_{p2}^3)(C_{22}e_{21}+D_2e_3)$$
$$\mu_2(e_2^3,t)=-D_3^{-1}D_2^{-1}\Delta G_0(e_2^3,t)(C_{20}+C_{22}z_{p21}+D_2z_{p3}) \tag{6.9}$$
$$\mu_3(e_{21},e_3)=-D_3^{-1}(C_{32}e_{21}+C_{33}e_3)$$

$$G_0(e_2^3+z_{p2}^3)=L_2^{-1}(e_2+z_{p2})\frac{\partial\Psi_3(e_2^3+z_{p2}^3)}{\partial e_2}=$$

$$= \left\| \begin{matrix} -k_{f11} & -(e_{21}+z_{p21})(e_{32}+z_{p32}) \\ \dfrac{e_{32}+z_{p32}}{e_{21}+z_{p21}} - k_{f11} + & \dfrac{-F_{f1}+e_{31}+z_{p31}-k_{f12}(e_{32}+z_{p32})}{e_{21}+z_{p21}} \end{matrix} \right\| \quad (6.10)$$

where $z_{ik} = e_{ik} + z_{pik}$ $(i = 2, 3, k = 1, 2)$

$$L_2^{-1}(\Phi_2(e_{x2}+x_{p2})) = L_{2x}(e_{x2}+x_{p2}) = \left\| l_{2xij}(e_{x2}+x_{p2}) \right\|_{i,j=1,2} \equiv$$

$$\equiv L_2^{-1}(e_2+z_{p2}) = L_{2z}(e_2+z_{p2}) = \left\| l_{2zij}(e_2+z_{p2}) \right\|_{i,j=1,2} =$$

$$= \left\| \begin{matrix} \cos z_{22} & \sin z_{22} \\ -\dfrac{\sin z_{22}}{z_{21}} & \dfrac{\cos z_{22}}{z_{21}} \end{matrix} \right\|$$

$$\Delta G_0(e_2^3, t) = G_0(e_2^3 + z_{p2}^3) - G_0(z_{p2}^3) = \left\| \Delta g_{0ij}(e_2^3, t) \right\|_{i,j=1,2}$$

$$\Delta g_{011}(e_2^3, t) = 0, \quad \Delta g_{012}(e_2^3, t) = -(e_{21}+z_{p21})e_{32} - z_{p32}e_{21}$$

$$\Delta g_{021}(e_2^3, t) = (e_{21}+z_{p21})^{-1} z_{p21}^{-1}(z_{p21}e_{32} - z_{p32}e_{21}) \quad (6.11)$$

$$\Delta g_{022}(e_2^3, t) = (e_{21}+z_{p21})^{-1} z_{p21}^{-1}\left[z_{p21}e_{31} - k_{f12}z_{p21}e_{32} + (F_{f1} - z_{p31} + k_{f12}z_{p32})e_{21}\right]$$

$$\Delta N_i(e_{x2}, t) = \Delta L_{xi}(e_{x2}, t) = L_i^{-1}(\Phi_2(e_{x2}+x_{p2})) - L_i^{-1}(\Phi_2(x_{p2})) =$$

$$= D_{i-1}^{-1}\Delta L_{x,i-1}(e_{x2}, t) \equiv D_{i-1}^{-1}\Delta L_{z,i-1}(e_2, t), \quad i = 3, 4 \quad (6.12)$$

$$\Delta L_{2x}(e_{x2}, t) = \left\| \Delta l_{2xij}(e_{x2}, t) \right\|_{i,j=1,2} = L_2^{-1}(\Phi_2(e_{x2}+x_{p2})) -$$

$$- L_2^{-1}(\Phi_2(x_{p2})) = L_2^{-1}(e_2+z_{p2}) - L_2^{-1}(z_{p2}) \equiv \Delta L_{2z}(e_2, t) = \left\| \Delta l_{2zij}(e_2, t) \right\|_{i,j=1,2} \quad (6.13)$$

$$\Delta l_{2xij}(e_{x2}, t) = l_{2xij}(e_{x2}+x_{p2}) - l_{2xij}(x_{p2}) \equiv \Delta l_{2zij}(e_2, t) =$$

$$= l_{2zij}(e_2+z_{p2}) - l_{2zij}(z_{p2}), \quad i, j = 1, 2$$

$$\Delta l_{2z11}(e_2, t) = \cos(e_{22}+z_{p22}) - \cos z_{p22}, \quad \Delta l_{2z12}(e_2, t) = \sin(e_{22}+z_{p22}) - \sin z_{p22},$$

$$\Delta l_{2z21}(e_2, t) = -\frac{\sin(e_{22}+z_{p22})}{e_{21}+z_{p21}} + \frac{\sin z_{p22}}{z_{p21}},$$

$$\Delta l_{2z22}(e_2, t) = \frac{\cos(e_{22}+z_{p22})}{e_{21}+z_{p21}} - \frac{\cos z_{p22}}{z_{p21}} \quad (6.14)$$

Let us estimate the modulus $|\Delta L_{2z}(e_2, t)|$ of the matrix function ΔL_{2z} (6.13), (6.14). We have

$$|\Delta L_{2z}(e_2, t)| \leq \sum_{i=1}^{2}\sum_{j=1}^{2} |\Delta l_{2zij}(e_2, t)| \leq \sum_{i=1}^{2}\sum_{j=1}^{2} \left| \left[\int_0^1 J_{1ij}(\theta e_2, t) d\theta \right] e_2 \right| \leq$$

$$\leq 2\varepsilon_V^{-2}|e_{21}| + 2(1+\varepsilon_V^{-1})|e_{22}| \leq k_{\Delta L2z}(|e_{21}| + |e_{22}|), \quad t \geq t_0 \quad (6.15)$$

where

$$J_{1ij}(e_2,t) = \left\|J_{1ij1}(e_2,t), J_{1ij2}(e_2,t)\right\| = \frac{\partial \Delta l_{2zij}(e_2,t)}{\partial e_2}$$

$$i=1,2;\; j=1,2;\; J_{111}(e_2,t) = \left\|0, -\sin(e_{22}+z_{p22})\right\|$$

$$J_{112}(e_2,t) = \left\|0, \cos(e_{22}+z_{p22})\right\|$$

$$J_{1211}(e_2,t) = \frac{\sin(e_{22}+z_{p22})}{(e_{21}+z_{p21})^2},\quad J_{1212}(e_2,t) = -\frac{\cos(e_{22}+z_{p22})}{e_{21}+z_{p21}}$$

$$J_{1221}(e_2,t) = -\frac{\cos(e_{22}+z_{p22})}{(e_{21}+z_{p21})^2},\quad J_{1222}(e_2,t) = -\frac{\sin(e_{22}+z_{p22})}{e_{21}+z_{p21}}$$

$$\sup_{\tilde e_2\in[0,e_2],\; t\geqslant t_0} \left|J_{11j}(\tilde e_2,t)\right| = 1,\quad \sup_{\tilde e_2\in[0,e_2],\; t\geqslant t_0} \left|J_{12i1}(\tilde e_2,t)\right| = \varepsilon_V^{-2}$$

$$\sup_{\tilde e_2\in[0,e_2],\; t\geqslant t_0} \left|J_{12i2}(\tilde e_2,t)\right| = \varepsilon_V^{-1} \quad (i=1,2;\; j=1,2) \tag{6.16}$$

$$[0,e_2] = \left\{\bar e_2 | \bar e_2 = \theta e_2,\; e_2+z_{p2}=z_2\in\Omega_{\Psi_2} \quad 0\leqslant\theta\leqslant 1\right\}$$

$$k_{\Delta L2z} = \max\left\{2\varepsilon_V^{-2}, 2(1+\varepsilon_V^{-1})\right\}$$

Taking relations (6.2), (6.5), (6.12)–(6.16) into account, we obtain

$$\begin{aligned}
|e_3(t)| &\leqslant \left|\Delta M_3(e_{x2}^2,t)\right| + \left|\Delta N_3(e_{x2},t)x_{p3}\right| + \left|N_3(e_{x2}+x_{p2})e_{x3}\right| \leqslant \\
&\leqslant \left|-D_2^{-1}C_{22}e_{21}\right| + \left|D_2^{-1}\Delta L_2^{-1}(e_{x2},t)x_{p3}\right| + \left|D_2^{-1}L_2^{-1}(e_{x2}+x_{p2})e_{x3}\right| \leqslant \\
&\leqslant \nu_{30}\left[|e_{21}(t)| + k_{\Delta L2z}(|e_{21}(t)|+|e_{22}(t)|) + \nu_2|e_{x3}(t)|\right] \leqslant \\
&\leqslant \nu_{32}(|e_{21}(t)|+|e_{22}(t)|+|e_{x3}(t)|) \leqslant 2\nu_{32}|e_2(t)| + \nu_{32}|e_{x3}(t)| \leqslant \\
&\leqslant \nu_3(|e_{x2}(t)|+|e_{x3}(t)|),\quad t\geqslant t_0
\end{aligned} \tag{6.17}$$

where

$$\nu_{30} = \max\left\{\left|D_2^{-1}\right|\left|C_{22}\right|,\; \left|D_2^{-1}\right|k_{xp3},\; \left|D_2^{-1}\right|\right\},$$

$$k_{xp3} = \sup_{t\geqslant t_0}|x_{p3}(t)| = \sup_{t\geqslant t_0}\left|\Psi_3(z_{p2}^3(t))\right|,\quad \nu_{31} = \max\{1+k_{\Delta L2z}, k_{\Delta L2z}, \nu_2\} \tag{6.18}$$

$$\nu_{32} = \nu_{30}\nu_{31},\quad \nu_3 = \max\{2\nu_{32}\nu_2, \nu_{32}\}$$

Let us estimate $|e_4(t)|$ (6.7). To do this we first estimate the modulus $|G_0(e_2^3 + z_{p2}^3)|$ of the matrix function G_0 (6.10). We obtain

$$\left|G_0(e_2^3+z_{p2}^3)\right| \leqslant \sum_{i=1}^{2}\sum_{j=1}^{2}\left|g_{0ij}(e_2^3+z_{p2}^3)\right| \leqslant \left|k_{f11}\right| +$$

$$+\left|(e_{21}+z_{p21})(e_{32}+z_{p32})\right| + \left|(e_{21}+z_{p21})^{-1}(e_{32}+z_{p32})\right| +$$

$$+\left|-k_{f11}+\left(e_{21}+z_{p21}\right)^{-1}\left[-F_{f1}+e_{31}+z_{p31}-k_{f12}\left(e_{32}+z_{p32}\right)\right]\right|\leq$$

$$\leq k_{001}+k_{031}\left|e_{31}+z_{p31}\right|+k_{032}\left|e_{32}+z_{p32}\right|\leq k_{0G0}\left(1+\left|e_{31}+z_{p31}\right|+\left|e_{32}+z_{p32}\right|\right)\leq$$

$$\leq k_{G0}\left(1+\left|e_{31}\right|+\left|e_{32}\right|\right),\quad t\geq t_0 \tag{6.19}$$

where

$$k_{001}=2k_{f11}+\left|F_{f1}\right|/\varepsilon_V,\quad k_{031}=\sup_{z_{21}\in\Omega_{z21},\ t\geq t_0}\left|z_{21}^{-1}(t)\right|=\varepsilon_V^{-1}$$

$$k_{032}=\sup_{z_{21}\in\Omega_{z21},\ t\geq t_0}\left[\left|z_{21}(t)\right|+\left(1+k_{f12}\right)\left|z_{21}^{-1}(t)\right|\right]=k_V+\left(1+k_{f12}\right)\varepsilon_V^{-1} \tag{6.20}$$

$$k_{0G0}=\max\left\{k_{001},k_{031},k_{032}\right\},\quad k_{G0}=k_{0G0}\left(1+k_{zp31}+k_{zp32}\right)$$

We will now estimate the modulus $\Delta G_0(e_2^3, t)$ for the matrix function ΔG_0 (6.11). We have

$$\left|\Delta G_0\left(e_2^3,t\right)\right|\leq\left|\Delta g_{012}\left(e_2^3,t\right)\right|+\left|\Delta g_{021}\left(e_2^3,t\right)\right|+\left|\Delta g_{022}\left(e_2^3,t\right)\right|\leq$$

$$\leq k_{121}\left|e_{21}\right|+k_{131}\left|e_{31}\right|+k_{132}\left|e_{32}\right|\leq k_{\Delta G0}\left(\left|e_{21}\right|+\left|e_{31}\right|+\left|e_{32}\right|\right),\quad t\geq t_0 \tag{6.21}$$

where

$$k_{121}=\sup_{t\geq t_0}\left[\left|z_{p32}(t)\right|+\left|\left(e_{21}(t)+z_{p21}(t)\right)^{-1}z_{p21}^{-1}(t)z_{p32}(t)\right|+\right.$$

$$\left.+\left|\left(e_{21}(t)+z_{p21}(t)\right)^{-1}z_{p21}^{-1}(t)\left(F_{f1}-z_{p31}(t)+k_{f12}z_{p32}(t)\right)\right|\right]=$$

$$=k_{zp3}+\varepsilon_V^{-2}\left[\left|F_{f1}\right|+k_{zp31}+\left(1+k_{f12}\right)k_{zp32}\right]$$

$$k_{131}=\sup_{t\geq t_0}\left|\left(e_{21}(t)+z_{p21}(t)\right)^{-1}\right|=k_{031}=\varepsilon_V^{-1} \tag{6.22}$$

$$k_{132}=\sup_{t\geq t_0}\left[\left|e_{21}(t)+z_{p21}(t)\right|+\left|\left(e_{21}(t)+z_{p21}(t)\right)^{-1}\left(1+k_{f12}\right)\right|\right]=$$

$$=k_V+\varepsilon_V^{-1}\left(1+k_{f12}\right),\quad k_{\Delta G0}=\max\left\{k_{121},k_{131},k_{132}\right\}$$

Let us estimate the modulus $\Delta M_4(e_2^3, t)$ of the vector function ΔM_4 (6.9). Using estimates (6.19), (6.20) and (6.21), (6.22) for the matrix functions G_0 (6.10) and ΔG_0 (6.11), we obtain

$$\left|\Delta M_4\left(e_{x2}^3,t\right)\right|\leq\left|\mu_1\left(e_2^3,t\right)\right|+\left|\mu_2\left(e_2^3,t\right)\right|+\left|\mu_3\left(e_{21},e_3\right)\right|\leq$$

$$\leq k_{\mu1}\left(1+\left|e_{31}\right|+\left|e_{32}\right|\right)\left(\left|e_{21}\right|+\left|e_3\right|\right)+k_{\mu2}\left(\left|e_{21}\right|+\left|e_{31}\right|+\left|e_{32}\right|\right)+$$

$$+k_{\mu3}\left(\left|e_{21}\right|+\left|e_3\right|\right)\leq k_{\Delta M4}\left[\left(1+\left|e_{31}\right|+\left|e_{32}\right|\right)\left(\left|e_{21}\right|+\left|e_3\right|\right)+\right.$$

$$\left.+2\left|e_{21}\right|+\left|e_{31}\right|+\left|e_{32}\right|+\left|e_3\right|\right],\quad t\geq t_0 \tag{6.23}$$

where

$$\left|\mu_1\left(e_2^3,t\right)\right|=\left|-D_3^{-1}D_2^{-1}G_0\left(e_2^3+z_{p2}^3\right)\left(C_{22}e_{21}+D_2e_3\right)\right|\leq$$

$$\leqslant \left|-D_3^{-1}D_2^{-1}\right|\left|G_0\left(e_2^3+z_{p2}^3\right)\right|\left|C_{22}e_{21}+D_2e_3\right|\leqslant$$
$$\leqslant k_{\mu 1}\left(1+|e_{31}|+|e_{32}|\right)\left(|e_{21}|+|e_3|\right),\quad t\geqslant t_0$$
$$k_{\mu 1}=\left|D_3^{-1}D_2^{-1}\right|k_{G0}k_{0\mu 1},\quad k_{0\mu 1}=\max\left\{|C_{22}|,|D_2|\right\}$$
$$\left|\mu_2\left(e_2^3,t\right)\right|=\left|-D_3^{-1}D_2^{-1}\Delta G_0\left(e_2^3,t\right)\left(C_{20}+C_{22}z_{p21}+D_2z_{p3}\right)\right|\leqslant$$
$$\leqslant k_{\mu 2}\left(|e_{21}|+|e_{31}|+|e_{32}|\right),\quad t\geqslant t_0 \tag{6.24}$$
$$k_{\mu 2}=\left|D_3^{-1}D_2^{-1}\right|k_{\Delta G0}\left|k_{0\mu 2}\right.,$$
$$\sup_{t\geqslant t_0}\left(|C_{20}|+|C_{22}|\left|z_{p21}(t)\right|+|D_2|\left|z_{p3}(t)\right|\right)\leqslant |C_{20}|+|C_{22}|k_V+|D_2|k_{zp3}=k_{0\mu 2}$$
$$\left|\mu_3(e_{21},e_3)\right|=\left|-D_3^{-1}(C_{32}e_{21}+C_{33}e_3)\right|\leqslant k_{\mu 3}\left(|e_{21}|+|e_3|\right)$$
$$k_{\mu 3}=\max\left\{\left|D_3^{-1}C_{32}\right|,\left|D_3^{-1}C_{33}\right|\right\},\quad k_{\Delta M4}=\max\left\{k_{\mu 1},k_{\mu 2},k_{\mu 3}\right\}$$

Using relations (6.5), (6.19)–(6.24), we estimate the modulus $|e_4(t)|$ (6.7). We obtain

$$\left|e_4(t)\right|\leqslant\left|\Delta M_4\left(e_{x2}^3,t\right)\right|+\left|\Delta N_4\left(e_{x2},t\right)x_{p4}\right|+\left|N_4\left(e_{x2}+x_{p2}\right)e_{x4}\right|\leqslant$$
$$\leqslant k_{\Delta M4}\left[\left(1+|e_{31}|+|e_{32}|\right)\left(|e_{21}|+|e_3|\right)+2|e_{21}|+|e_{31}|+|e_{32}|+|e_3|\right]+$$
$$+k_{\Delta N4}|e_2|+k_{N4}|e_{x4}|\leqslant k_{e4}\left[3|e_{21}|+|e_{31}|+|e_{32}|+\right.$$
$$+2|e_3|+\left(|e_{31}|^2+|e_{21}|^2\right)/2+\left(|e_{32}|^2+|e_{21}|^2\right)/2+$$
$$\left.+\left(|e_3|^2+|e_{31}|^2\right)/2+\left(|e_3|^2+|e_{32}|^2\right)/2+|e_2|+|e_{x4}|\right]\leqslant$$
$$\leqslant k_{e4}\left(4|e_2|+4|e_3|+|e_2|^2+2|e_3|^2+|e_{x4}|\right)\leqslant$$
$$\leqslant \nu_4\left(|e_{x2}|+|e_{x3}|+|e_{x4}|+|e_{x2}|^2+|e_{x3}|^2\right),\quad t\geqslant t_0 \tag{6.25}$$

where

$$k_{\Delta N4}=2\left|D_3^{-1}D_2^{-1}\right|k_{\Delta L2z}k_{xp4},$$
$$k_{xp4}=\sup_{t\geqslant t_0}\left|x_{p4}(t)\right|=\sup_{t\geqslant t_0}\left|\Psi_4\left(z_{p2}^4(t)\right)\right|$$
$$k_{N4}=\left|D_3^{-1}D_2^{-1}\right|\nu_2,\quad k_{e4}=\max\left\{k_{\Delta M4},k_{\Delta N4},k_{N4}\right\} \tag{6.26}$$
$$\nu_4=k_{e4}\times\max\left\{4\nu_2,4\nu_3,\nu_2^2,2\nu_3^2,1\right\}$$

This research was partially supported by the Russian Foundation for Basic Research (98-01-00581).

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Translated by D.L